Biharmonic Problems with Steklov-type and Farwig Boundary Conditions and their Applications

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Abstract: We study some properties of solutions of biharmonic problems with Steklov-type and Farwig boundary conditions and their application in technique and engineering. Using the scattering model, to solve these biharmonic problems, which have applications in particular in radar imaging, we need to solve the Dirichlet and Neumann boundary value problems for the Poisson equation.

Key words: Biharmonic operator, boundary value problems, scattering model, variational method.

1. Introduction

Let $\Omega \subset R^n$, $n \geq 2$, be a bounded Lipschitz domain with connected boundary $\partial\Omega$, and $\Omega \cup \partial\Omega = \Omega$ is the closure of Ω. We consider the following boundary value problems for the biharmonic equation in Lipschitz domains:

$$\Delta^2 u = f, \qquad x \in \Omega \tag{1}$$

with the Steklov-type boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} = g_1 & \text{on} \quad \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} + \tau\, u = g_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

or the Farwig boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} = h_1 & \text{on} \quad \partial\Omega, \\ \frac{\partial \Delta u}{\partial \nu} = h_2 & \text{on} \quad \partial\Omega, \end{cases} \tag{3}$$

where ν is the outer unit normal vector to the domain with the Lipschitz boundary $\partial\Omega$, $\tau \in C(\partial\Omega)$, $\tau \geq 0$, $\tau \not\equiv 0$.

Elliptic problems with parameters in the boundary conditions are called Steklov (or Steklov-type) problems from their first appearance in [31]. In the case of the biharmonic operator, these conditions were first considered in [3], [10] and [29], who studied the isoperimetric properties of the first eigenvalue.

The main novelty of this paper is the consideration of a boundary condition with a parameter, specifically a Steklov-type boundary condition. Steklov and Steklov-type boundary value problems with parameters are understudied and are of great importance in the development of applications in mechanical engineering and technology, as well as in physical chemistry, medicine, and elsewhere.

In the paper, by solving the Steklov-type biharmonic problem and deriving a mathematical model of scattering, we were able to describe, in particular, the radar process.

The standard elliptic regularity results are available in [7]. This monograph covers higher order linear and nonlinear elliptic boundary

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value problems, mainly with the biharmonic (polyharmonic) operator as leading principal part. Underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and L^p and Schauder estimates, the focus is on positivity.

In [6], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted L^q-space the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

We note paper [2], which explains for the biharmonic operator that the higher the order, the greater the variety of possible boundary value problems and associated hybrid Green's functions. The advantage of convoluted higher order Green's functions is that they allow to decompose the boundary value problem for a linear higher order Poisson equation into some for a system of first order Poisson equations.

In [4], a weak solution of a mixed boundary value problem for a biharmonic equation in the plane is studied, in which, using the Green formula, the problem is transformed into a system of Fredholm integral equations for unknown data on different parts of the boundary. The existence and uniqueness of solutions of the system of boundary integral equations in the corresponding Sobolev spaces are also established.

Boundary value problems for a biharmonic (polyharmonic) equation in unbounded domains by the author are studied in [12] [28], in which the condition of the boundedness of the following weighted Dirichlet integral of solution is finite, namely

$$\int_{\Omega} |x|^a \sum_{|\alpha|=2} |\partial^\alpha u|^2 \, dx < \infty,$$

where a∈ R is a fixed number and

$$\sqrt{\sum_{|\alpha|=2} |\partial^\alpha u|^2}$$

denotes the Frobenius norm of the Hessian matrix of u. In particular, in [12] [28] has been studied the dimension of the space of the solutions to the boundary value problems for a biharmonic (polyharmonic) equation, providing explicit formulas which depends on n and a.

In [5], [8] and [9], generalizations of the Hardy inequality were established for bounded and for a wide class of unbounded domains, and applied these to investigate boundary value problems for elliptic equations and systems. In particular, the problems of the existence, the uniqueness, the stability and the asymptotic expansions of solutions of boundary value problems were studied.

Notation: $C^\infty_0(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω; $H^m(\Omega)$ is the Sobolev space obtained by the completion of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u(x); H^m(\Omega)\| = \left(\int_{\Omega} \sum_{|\alpha|\le m} |\partial^\alpha u(x)|^2 dx \right)^{1/2}, \; m = 1, 2,$$

where $\partial\alpha \equiv \partial|\alpha|/\partial x\ \alpha\_1 \ldots \partial x\ \alpha\_n$, $\alpha = (\alpha\_1, \ldots, \alpha\_n)$ is a multi-index, $\alpha\_i \geq 0$ are

$$\|u(x); H^m(\Omega \cap B_0(R))\| = \left(\int_{\Omega \cap B_0(R)} \sum_{|\alpha| \le m} |\partial^\alpha u(x)|^2\, dx \right)^{1/2}$$

integers, and $|\alpha| = \alpha\_1 + \cdots + \alpha\_n$; $H^m(\Omega)$ is the space obtained by the completion of $C^\infty(\Omega)$ with respect to the norm for all open balls $B_0(R) := \{x: |x| < R\}$ in R^n for which $\Omega \cap B_0(R) /= \emptyset$.

Finally, $H^{1/2}(\partial\Omega)$ is the usual trace space on the boundary and $H^{-1/2}(\partial\Omega)$ is its dual (see, for ex., [1]).

Definition 1. A solution of the biharmonic equation (1) in Ω is a function $u \in H^2(\Omega)$ such that, for every function $\phi \in C\infty(\Omega)$, the following integral identity holds:

$$\int_\Omega \Delta u\, \Delta\varphi\, dx = \int_\Omega f\, \varphi\, dx. \tag{4}$$

Definition 2. A function u is a solution of the Steklov-type problem (1), (2), with g_1 = 0, g_2 = 0, if $u \in H^2(\Omega),\ \partial u/\partial\nu = 0\ on\ \partial\Omega$, such that for every function $\phi \in C^0_\infty(\mathbb{R}^n),\ \partial\phi/\partial n = 0\ on\ \partial\Omega$, the following integral identity holds

$$\int_\Omega \Delta u\, \Delta\varphi\, dx - \int_{\partial\Omega} \tau\, u\, \varphi\, ds = \int_\Omega f\, \varphi\, dx. \tag{5}$$

Definition 3. A function u is a solution of the Farwig problem (1), (3) with $h_1 = h_2 = 0$, if $u \in \overset{\circ}{H}{}^2(\Omega),\ \partial u/\partial\nu = 0\ on\ \partial\Omega$, and the integral identity (4) holds for every function $\varphi \in \overset{\circ}{H}{}^2(\mathbb{R}^n)$ such that $\partial\phi/\partial\nu = 0$ on $\partial\Omega$.

2. A Scattering Model

In the section we derive the mathematical model used for describing the radar process. In our parametrization the unknown is the height function H. As will be shown the height function is determined in two steps. In the first step L(H), with L a certain second-order differential operator, is determined. After retrieving H, the equation L(H) = f must be solved. To a good approximation the operator L can be replaced by the Laplacian. So, the second step simply consists of solving the Poisson equation over some smooth bounded domain, usually a rectangular region in the plane. The problem here is that no natural boundary conditions are available.

Here we will brie y discuss the mathematical inverse problem to be resolved in order to recover the ground topography height function from radar data. First cylindrical coordinates **(r, ϕ, z)** are introduced according to Fig. 1, where it is understood that the aircraft is flying at a constant speed along the **z**-axis. Further **r** denotes the distance from a point on the ground surface to the **z**-axis and ϕ is the angle between radius vector and a horizontal plane through the **z**-axis. Then the ground surface may be described by a function H(r, z) through the equation

$$\frac{H(r,z)}{r} - \varphi = 0. \tag{6}$$

When **r** is large, H(r, z) is approximately a Cartesian height function. Fig. 2 shows a top view of the same scene. We have also indicated an aspect vector from the aircraft to some point on the ground, forming an angle **θ** with a vertical plane through the aircraft. Normalized to unit length, the aspect vector is denoted by $\hat{n}$ Accordingly

$$\hat{n} = \cos\theta\,\hat{r}(\varphi) + \sin\theta\,\hat{z}. \tag{7}$$

Here $\hat{r}(\phi)$ denotes the cylindrical unit basis vector corresponding to the **r**-coordinate for the ground point as shown in the Fig. 2. For a point on the ground surface with coordinates (r, ϕ, z) we obtain, from Eq. (6), the following expression for the ground surface normal $\bar{m}$,

$$\bar{m} = grad\left(\frac{H(r,z)}{r} - \varphi\right) = \frac{\partial(H/r)}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial H}{\partial z}\hat{z} - \frac{1}{r}\hat{\varphi}. \tag{8}$$

Let $\hat{m}$ denote the normalized normal. Then

$$\hat{m}\circ\hat{n} = \left(r\cos\theta\frac{\partial(H/r)}{\partial r} + \sin\theta\frac{\partial H}{\partial z}\right)\Big/\sqrt{1 + r^2\left(\frac{\partial(H/r)}{\partial r}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2}. \tag{9}$$

Note that **(r, ϕ, z)** in Eq. (9) are related to the ground surface point and not to the position of the aircraft.

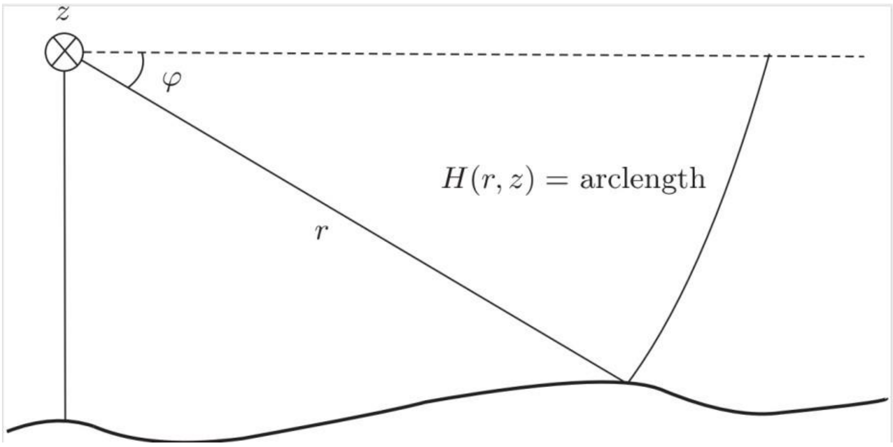

Figure 1. The ground surface measured at a fixed aircraft position.

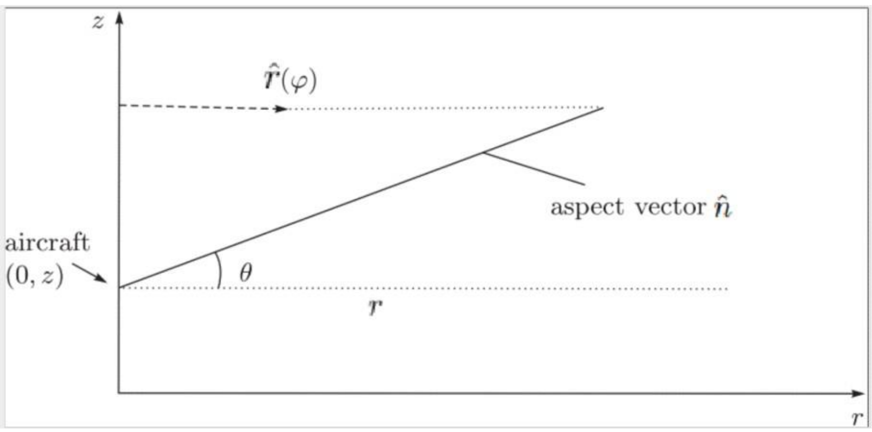

Figure 2. The measuring geometry as seen from above.

Let $(z_0, 0)$ be a position of the aircraft and **R** the distance to some point on the surface. According to Fig. 3 the coordinates (r, z) are then equal to $(R\cos\theta, z_0 + R\sin\theta)$. Next, to obtain a scattering model we will assume that the reflectivity from a ground surface element (see Fig. 4) is

$$\approx \frac{\hat{m}\circ\hat{n}}{R}\,dR\,d\theta. \tag{10}$$

From Fig. 4, where a vertical plane through $(z_0, 0)$ (the aircraft) and the ground point

$(R\cos\theta, z_0 + R\sin\theta)$ is displayed, we conclude that the solid angle dΩ under which the surface element dS is seen from the antenna is approximately

$$\frac{dR\,\cos\alpha\, Rd\,d\theta}{R^2} = -\frac{\hat{m}\circ\hat{n}}{R}dR\,d\theta.$$

In expression (10) we are consequently assuming that the local reflectivity is proportional to the solid angle occupied by the infinitesimal surface element dS. The total reflected signal $G(R,\ z_0)$ from all points at a distance **R** from the antenna may now be obtained by integration over the circle

$$C(R,z_0) = \{(r,z) : r^2 + (z-z_0)^2 = R^2\}$$

in Fig. 3.

$$G(R,z_0)dR = c\int_{-\pi}^{\pi}\frac{\hat{m}\circ\hat{n}(R\sin\theta, z_0 + R\cos\theta)}{R}d\theta dR$$

$$RG(R,z_0) = c\int_{-\pi}^{\pi}\hat{m}\circ\hat{n}(R\sin\theta, z_0 + R\cos\theta)d\theta. \tag{11}$$

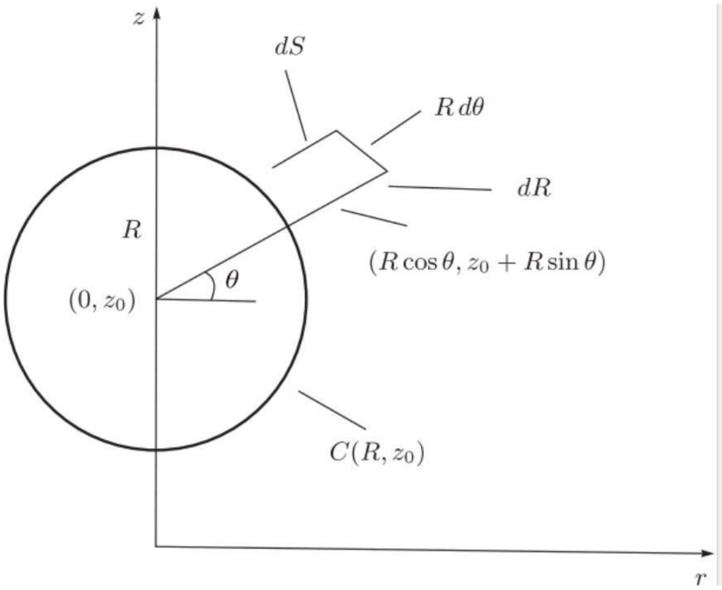

Figure 3. The coordinate system used to describe an infinitesimal surface element, d*S*.

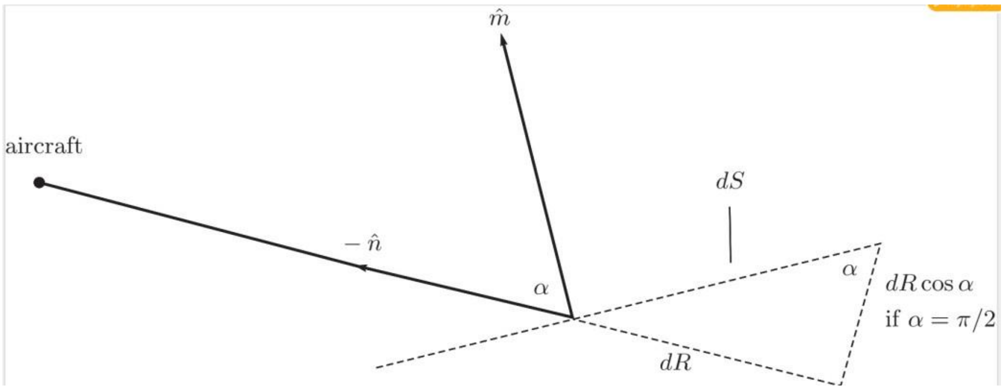

Figure 4. The infinitesimal surface element, dS, as it is seen from the aircraft.

Assuming that m̂ ∘ n̂ is small Eq. (9) may be replaced by

$$\hat{m}\circ\hat{n} = r\cos\theta\frac{\partial(H/r)}{\partial r} + \sin\theta\frac{\partial H}{\partial z}.$$

By inserting this into Eq. (11) we get, after multiplying by R,

$$R^2 G(R, z_0) = c \int_{-\pi}^{\pi} \left(rR\cos\theta \frac{\partial(H/r)}{\partial r} + R\sin\theta \frac{\partial H}{\partial z} \right) d\theta.$$

Using the parametrization

$$r = Rcos\theta, \quad z = z_0 + Rsin\theta,$$

this may be rewritten as a curve integral over

$C(R, z_0)$, with $dz = R\cos\theta d\theta$ and $dr = -R\sin\theta d\theta$,

$$R^2 G(R, z_0) = c \int_{C(R,z_0)} \left(r\frac{\partial(H/r)}{\partial r} dz - \frac{\partial H}{\partial z} dr \right). \tag{12}$$

By applying Green's formula, we get

$$R^2 G(R, z_0) = c \iint_{D(R,z_0)} \mathfrak{L}(H)(r, z)\, dz\, dr, \tag{13}$$

where D is the disc,

$$D(R, z_0) = \{(r, z) : r^2 + (z - z_0)^2 \leq R^2\}$$

And

$$\mathfrak{L}(H) = \frac{\partial}{\partial r}\left(r\frac{\partial(H/r)}{\partial r} \right) + \frac{\partial^2 H}{\partial z^2}. \tag{14}$$

The problem of finding the height function H from radar data $G(r, z)$ may now be divided into two parts:

(i) First solve the integral equation (13) for $L(H)(r, z) = f(r, z)$.

(ii) Next solve the partial differential equation

$$\mathfrak{L}(H) = f \tag{15}$$

for H. We note that if r is large and if $\hat{m} \circ \hat{n}$ is small it is reasonable to make the approximation

$$\mathfrak{L}(H) \approx \frac{\partial^2 H}{\partial r^2} + \frac{\partial^2 H}{\partial z^2} = \Delta H$$

so that Eq. (15) becomes Poisson's equation. To consider the first problem (i), both members in Eq. (13) are differentiated with respect to R. Then we get

$$\frac{1}{R}\frac{d}{dR}(R^2 G(R, z_0)) = c \int_{-\pi}^{\pi} \mathfrak{L}(H)(z_0 + R\cos\gamma, R\sin\gamma)\, d\gamma,$$

where the right-hand side is proportional to the average of $L(H)$ over the circle $C(R, z_0)$. Hence,

$$\mathfrak{L}(H)^{(F,F)}(\sigma, \omega) \sim |\omega| \left[\frac{1}{R}\frac{d}{dR}\{R^2 G(r, z)\} \right]^{(F,H_0)} (\sigma, \sqrt{\omega^2 + \sigma^2}). \tag{16}$$

H ere the notation (F, F) means that we have taken the Fourier transform with respect to both the variables and (F, H_0) means that we have taken Fourier transform with respect to the first variable and the Hankel-zero transform with respect to the second. After some calculations Eq. (16) may be rewritten

$$\mathfrak{L}(H)^{(F,F)}(\sigma, \omega) \sim |\omega| \sqrt{\omega^2 + \sigma^2} [RG(r, z)]^{(F,H_1)} (\sigma, \sqrt{\omega^2 + \sigma^2}). \tag{17}$$

Formula (17) may now be used in order to recover the function $L(H)$ in spatial coordinates.

Approximating $L(H)$ by ΔH we could rewrite Eq. (17) as

$$H^{(F,F)}(\sigma, \omega) \sim |\omega| \frac{1}{\sqrt{\omega^2 + \sigma^2}} [RG(r, z)]^{(F,H_1)} (\sigma, \sqrt{\omega^2 + \sigma^2}), \tag{18}$$

Where H_1 denotes that we have taken the Hankel-one transform with respect to the second variable. Then we could obtain H directly by a two-dimensional Fourier transform. However, our solution might be expected to have errors caused by, e.g. noisy radar data and errors caused by the particular numerical implementation of the inversion formula (16) (or Eq. (17)) and therefore we would rather prefer to divide the solution procedure into the two steps described above and to use the second step, the solution of

Poisson's equation, so that we perform some kind of regularization of the final solution. Note also that by using Eq. (18) as our solution formula we have tacitly assumed periodic boundary conditions for the Poisson equation.

3. Solution Concepts for the Poisson Equation

In this section we discuss different possibilities of defining a unique height function. Essentially our approach consists in minimizing some norm of the solution provided that it also satisfies the Poisson equation. In particular we consider the L^2- and H^1-norms. We also show how these two optimization problems may be reformulated as boundary value problems for the biharmonic equation.

In the domain Ω for the Poisson equation we consider the following boundary value problems

$$\Delta u = f, \quad x \in \Omega \tag{19}$$

with the Dirichlet boundary condition

$$u = g \quad \text{on} \quad \partial\Omega, \tag{20}$$

or the Neumann boundary conditions

$$\nabla u \cdot \nu = h \quad \text{on} \quad \partial\Omega, \tag{21}$$

where ν is the outer unit normal vector to $\partial\Omega$.

The boundary operators are independent of any particular choice of orientation for the rectangular coordinate systems. Finally, for Ω a rectangular region in, e.g., the plane

$$\Omega = \{(x, y) : a < x < b, c < y < d\},$$

there may be the following boundary conditions

$$u(a, y) = u(b, y), \quad u(x, c) = u(x, d),$$

or the periodic boundary conditions

$$\begin{cases} u(a,y) = u(b,y), & u(x,c) = u(x,d), \\ u_x(a,y) = u_x(b,y), & u_y(x,c) = u_y(x,d). \end{cases} \tag{23}$$

Provided g is smooth enough boundary conditions (20) de ne a unique solution of Eq. (19). For (21) and (23) the solution is determined up to a constant. It is also possible to use different mixtures of these three types of boundary conditions. Note that for cases (21) and (23) the following consistency conditions must hold, respectively:

$$\int_\Omega f\, dx = \int_{\partial\Omega} h\, ds \quad \text{and} \quad \int_\Omega f\, dx = 0.$$

We now consider a different way to select a solution to Eq. (19). Here we use a criterion function and optimize this criterion over the set of solutions to the Poisson equation. Scattering model of Section 2 shows the physical interpretation of function $u(x, y)$ is a surface function. We need to pick out the smoothest surface (in some sense) that fulfills Eq. (19), using the Sobolev space norms as criterion functions. Denote by $V_{f,i}$ the following set:

$$V_{f,i} = \{u \in H^i(\Omega) : \Delta u = f,\ f \in L^2(\Omega)\},\ i = 0, 1, 2, \tag{24}$$

where $H^0(\Omega) = L^2(\Omega)$.

The equality $\Delta u = f$ is to be interpreted in the sense of distributions. i.e.,

Definition 4. A solution of the Poisson equation (19) in Ω is a function $u \in H^1(\Omega)$ such that the following integral identity holds:

$$\int_\Omega u\, \Delta\varphi\, dx = \int_\Omega f\varphi\, dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Lemma 1. $V_{f,i}$ is a closed, convex and nonempty set of $H^i(\Omega)$.

Proof. The convexity is due to the linearity of Δ. To verify that $V_{f,i}$, $i = 0, 1, 2$, is nonempty it suffices to verify that $V_{f,2}$ is nonempty.

We assume that Ω $(0, 2\pi)^n$. Extend f by taking $f = 0$ in $(0, 2\pi)^n$ Ω. Then $V_{f,2}$ contains the function

$$u = f_0|x|^2/(2n) - \sum_{m\neq 0} e^{imx}/|m|^2$$

Assuming $f = \sum f_m e^{imx}$ and that m denotes a multi-index. To show that $V_{f,i}$ is closed we select a sequence $\{u_n\}_1^\infty \subset V_{f,i}$, such that $u_n \to u$ in $H^i(\Omega)$. Then $u_n \to u$ in L^2 and, by Cauchy's inequality

$$\left|\int_\Omega f\,\varphi\,dx - \int_\Omega u\,\Delta\varphi\,dx\right| = \left|\int_\Omega (u_n - u)\,\Delta\varphi\,dx\right| \le$$

$$\le \int_\Omega |u_n - u|^2\,dx \int_\Omega |\Delta\varphi|^2\,dx \to 0, \quad \forall\varphi \in C_0^\infty,$$

i.e. $\int_\Omega f\,\varphi\,dx = \int_\Omega u\,\Delta\varphi\,dx$ and $u \in V_{f,i}$

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Let α be a multi-index and $\beta_1 > 0$ a given parameter. We consider the following optimization problems:

$$I_0(u) \equiv \min_{u\in V_{f,0}} \int_\Omega |u|^2\,dx, \tag{25}$$

and

$$I_1(u) \equiv \min_{u\in V_{f,1}} \int_\Omega |u|^2\,dx + \beta_1 \int_\Omega \sum_{|\alpha|=1} |\partial^\alpha u|^2\,dx. \tag{26}$$

Theorem 1. Problems (25) and (26) have unique solutions u_0 and u_1, respectively.

Proof. The proof follows from Lemma 1 and the fact that we are minimizing Hilbert norms.

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By a standard variational method [30], u_0 $L^2(\Omega)$ solves problem (25) if and only if $\Delta u_0 = f$ and

$$\int_\Omega u_0\,\varphi\,dx = 0$$

for all $\phi \in L^2(\Omega)$ and $\Delta\phi = 0$.

Assume first that u_0 solves problem (25). Let v be defined as the unique solution of the Dirichlet problem,

$$\begin{cases} \Delta v = u_0 & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

If ϕ, $v \in H^1(\Omega)$ and $\Delta\phi$, $\Delta v \in L^2(\Omega)$, we have the Green formula

$$\int_\Omega \Delta v\,\varphi\,dx - \int_\Omega v\,\Delta\varphi\,dx = \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\varphi\,ds - \int_{\partial\Omega} v\,\frac{\partial\varphi}{\partial\nu}\,ds.$$

Now let $\phi \in H^1(\Omega)$ be a harmonic function, $\Delta\phi = 0$. Then we have

$$0 = \int_\Omega u_0\,\varphi\,dx = \int_\Omega \Delta v\,\varphi\,dx = \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\varphi\,ds - \int_{\partial\Omega} v\,\frac{\partial\varphi}{\partial\nu}\,ds + \int_\Omega v\,\Delta\varphi\,dx,$$

That is

$$\int_{\partial\Omega} \frac{\partial v}{\partial\nu}\varphi\,ds = 0 \quad \text{for all such} \quad \varphi.$$

Since there exists a unique function $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = f, \quad f \in L^2(\Omega) & \text{in} \quad \Omega, \\ u = g, \quad g \in H^{1/2}(\partial\Omega) & \text{on} \quad \partial\Omega, \end{cases}$$

and ϕ $_{\partial\Omega}$ may be chosen arbitrary in $H^{1/2}(\Omega)$, we conclude that $0 = (\nabla v \cdot v) \in H^{-1/2}(\Omega)$. We have proved that $u_0 = \Delta v \in L^2(\Omega)$, where v satisfies the Dirichlet biharmonic problem

$$\begin{cases} \Delta^2 v = f & \text{in} \quad \Omega, \\ v = \nabla v \cdot \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{27}$$

On the other hand, we claim that (27) cannot have more than one solution $v \in H^1(\Omega)$ with Δv

$\in L^2(\Omega)$. Indeed, assume that (27) is satisfied and consider the function $\psi \in L^2(\mathrm{R}^n)$ defined by

$$\psi = \begin{cases} v(x) & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

For arbitrary $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \psi \, \Delta\varphi \, dx = \int_\Omega v \, \Delta\varphi \, dx = \int_{\partial\Omega} v \frac{\partial\varphi}{\partial\nu} \, ds - \int_{\partial\Omega} \frac{\partial v}{\partial\nu} \varphi \, ds + \int_\Omega \varphi \, \Delta v \, dx,$$

i.e.

$$\int_{\mathbb{R}^n} \psi \, \Delta\varphi \, dx = \int_\Omega \varphi \, \Delta v \, dx \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

Let now $h \in C_0^\infty(\mathbb{R}^n)$ be dined by

$$h(x) = \begin{cases} \Delta v & \text{if} \quad x \in \Omega, \\ 0 & \text{if} \quad x \notin \Omega. \end{cases}$$

We have proved that

$$\Delta\psi = h$$

in the sense of distributions. Using the Fourier transformation, it follows that $\psi \in H^2(\mathrm{R}^n)$. Therefore, $v \in H^2(\Omega)$, and v must be the unique solution in $H^2(\Omega)$ of (27), being the unique minimizer in $\overset{\circ}{H}{}^2(\Omega)$ of the coercive quadratic functional

$$J(v) \equiv \int_\Omega \left(\frac{1}{2} |\Delta v|^2 - fv \right) dx.$$

Theorem 2. Let $u_1 = \Delta v$. For the solution u_1 of the problems (26), where $v \in H^2(\Omega)$ is the unique solution in the class $\{\psi \in H^1(\Omega): \Delta\psi \in H^1(\Omega)\}$ of the following biharmonic problem

$$\begin{cases} \Delta^2 v = f & in \quad \Omega, \\ v = \beta_1 \Delta v, \ \nabla v \cdot \nu = 0 & on \quad \partial\Omega. \end{cases} \tag{28}$$

Proof. By standard variational method, $u_1 \in H^1(\Omega)$ solves problem (26) if and only if $\Delta u_1 = f$ and

$$\int_\Omega (u_1 \varphi + \beta_1 \nabla u_1 \cdot \nabla\varphi) \, dx = 0$$

for all $\phi \in H^1(\Omega)$ and $\Delta\phi = 0$ in Ω.

Taking $\phi = 1$ we observe that

$$\int_\Omega u_1 \, dx = 0.$$

Let $v \in H^1(\Omega)$ be any solution of the Neumann problem

$$\begin{cases} \Delta v = u_1 & \text{in} \quad \Omega, \\ \nabla v \cdot \nu = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{29}$$

Applying Green's formula we have,

$$0 = \int_\Omega \varphi \, \Delta v \, dx + \beta_1 \int_\Omega \nabla(\Delta v) \nabla\varphi \, dx =$$

$$= \int_\Omega v \, \Delta\varphi \, dx + \int_{\partial\Omega} \varphi \frac{\partial v}{\partial\nu} \, ds - \int_{\partial\Omega} v \frac{\partial\varphi}{\partial\nu} \, ds + \beta_1 \int_{\partial\Omega} \Delta v \frac{\partial\varphi}{\partial\nu} \, ds - \beta_1 \int_\Omega \Delta v \, \Delta\varphi \, dx,$$

i.e.

$$\int_{\partial\Omega} (v - \beta_1 \Delta v)(\nabla\varphi \cdot \nu) \, ds = 0$$

for all $\phi \in H^1(\Omega)$ and $\Delta\phi = 0$ in Ω.

Since $\nabla\phi \cdot \nu \in H^{-1/2}(\partial\Omega)$ may be chosen arbitrarily apart from the condition

$$\int_{\partial\Omega} \nabla\varphi \cdot \nu \, ds = 0,$$

it follows that for some C = const

$$v - \beta_1 \Delta v = C \quad \text{on} \quad \partial\Omega.$$

Now the solution v is uniquely de ned up to an additive constant. This constant may □ be chosen so that $C = 0$.

■

We conclude this section by a theorem relating the solution of problems (25) and (26). First, we recall the following definition.

Definition 5. $\Omega \subset \mathrm{R}^n$ is called star-shaped if there exists $x_0 \in \Omega$ such that for all $x \in \Omega$ the set $\{t \in \mathrm{R}: \quad x_0 + t(x - x_0) \in \Omega\}$ is an interval.

Theorem 3. Assume that $\Omega \subset R^n$ is open, bounded and star-shaped. If $u_{1,\beta} \in H^1(\Omega)$ denotes the solution of problem (26) with the parameter $\beta_1 > 0$, and if u_0 $L^2(\Omega)$ denotes the solution of problem (25), then

$$u_{1,\beta_1} \to u_0 \quad in \quad L^2(\Omega) \quad as \quad \beta_1 \to 0+.$$

Proof. For $0 < \lambda < 1$ and x_0 chosen as in the previous definition, we take

$$\Omega_\lambda = \{x \in \mathbb{R}^n : \ x_0 + \lambda(x - x_0) \in \Omega\},$$

$$u_{0,\lambda}(x) = u_0(x_0 + \lambda(x - x_0)), \quad f_\lambda = f(x_0 + \lambda(x - x_0)).$$

Then [11],

$$\Delta u_{0,\lambda} = f_\lambda \quad \text{in} \quad \Omega_\lambda, \quad \Omega_\lambda \supset \overline{\Omega}, \quad u_{0,\lambda} \in H^2_{loc}(\Omega_\lambda).$$

Since $H^2(\Omega_\lambda) \supset H^2(\Omega)$, it follows that $u_{0,\lambda} \in H^2(\Omega)$. Further it is rather easy to see that

$$\int_\Omega |u_{0,\lambda} - u_0|^2 \, dx \to 0,$$

And

$$\int_\Omega |f_\lambda - f|^2 \, dx \to 0 \quad \text{as} \quad \lambda \to 1.$$

Next define $v_\lambda \in \overset{\circ}{H}{}^1(\Omega)$ by

$$\Delta v_\lambda = f - f_\lambda \quad \text{in} \quad \Omega_\lambda.$$

Then

$$\int_\Omega |v_\lambda|^2 \, dx \leq ||v_\lambda||_{H^1(\Omega)} \leq C \int_\Omega |f - f_\lambda|^2 \, dx.$$

Consequently, taking $w_\lambda = u_{0,\lambda} + v_\lambda$, we have first,

$$w_\lambda \in H^1(\Omega), \quad \Delta w_\lambda = f \quad \text{in} \quad \Omega,$$

And hence,

$$\int_\Omega |w_\lambda - u_0|^2 \, dx \to 0 \quad \text{as} \quad \lambda \to 1.$$

Now, if $\varepsilon > 0$ is given, we may choose a λ close enough to 1, so that

$$\int_\Omega w_\lambda^2 \, dx < \int_\Omega u_0^2 \, dx + \varepsilon/2.$$

Further, by definition

$$\int_\Omega u_{1,\beta_1}^2 \, dx + \beta_1 \int_\Omega |\nabla u_{1,\beta_1}|^2 \, dx \leq \int_\Omega w_\lambda^2 \, dx + \beta_1 \int_\Omega |\nabla w_\lambda|^2 \, dx.$$

Since

$$||w_\lambda||_{H^1(\Omega)} \leq C \int_\Omega |f|^2 \, dx$$

we have, for sufficiently small β_1.

$$\int_\Omega u_{1,\beta_1}^2 \, dx + \beta_1 \int_\Omega |\nabla u_{1,\beta_1}|^2 \, dx \leq \int_\Omega u_0^2 \, dx + \varepsilon.$$

It follows that

$$\lim_{\beta_1 \to 0+} \sup \int_\Omega |u_{1,\beta_1}|^2 \, dx \leq \int_\Omega |u_0|^2 \, dx.$$

Further, for some sub sequence of β_1, we have

$$u_{1,\beta_1} \to \widetilde{u} \quad \text{in} \quad L^2(\Omega) \quad (\text{weakly}),$$

$$\Delta \widetilde{u} = f \quad \text{in} \quad \Omega,$$

And

$$\int_\Omega |\widetilde{u}|^2 \, dx \leq \lim_{\beta_1 \to 0+} \inf \int_\Omega |u_{1,\beta_1}|^2 \, dx.$$

But then $\Delta \tilde{u} = f$ and

$$\int_\Omega |\widetilde{u}|^2 \, dx \leq \int_\Omega |u_0|^2 \, dx$$

which, by definition of u_0, implies that $\tilde{u} = u_0$. So,

$$u_{1,\beta_1} \to u_0 \quad \text{in} \quad L^2(\Omega) \quad (\text{weakly}).$$

Next

$$\lim_{\beta_1\to 0^+}\sup\int_\Omega |u_{1,\beta_1}-u_0|^2\,dx$$
$$=\lim_{\beta_1\to 0^+}\sup\int_\Omega |u_{1,\beta_1}|^2\,dx-2\lim_{\beta_1\to 0^+}\int_\Omega u_{1,\beta_1}\,u_0\,dx+\int_\Omega |u_0|^2\,dx$$
$$\le\int_\Omega |u_0|^2\,dx-2\int_\Omega |u_0|^2\,dx+\int_\Omega |u_0|^2\,dx=0.$$

Finally, since this strong limit u_0 is uniquely defined we may conclude, by a standard argument that

$$u_{1,\beta_1}\to u_0 \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad \beta_1\to 0^+$$

without restriction to any sub-sequence.

▪

Remark 1. All convex sets are star-shaped. Rectangles Ω appearing in our applications are thus star-shaped.

4. Conflict Of Interest

The author of this paper declares that he has no conflicts of interest.

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